1 $\mathbf{AX} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + -2 \times -1 \\ -1 \times 2 + 3 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \times 2 + 2 \times -1 \\ 1 \times 2 + 1 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{AY} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 1 + -2 \times 3 \\ -1 \times 1 + 3 \times 3 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + 0 \times -1 \\ 0 \times 2 + 1 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{AC} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 2 + -2 \times 1 & 1 \times 1 + -2 \times 1 \\ -1 \times 2 + 3 \times 1 & -1 \times 1 + 3 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 1 + 1 \times -1 & 2 \times -2 + 1 \times 3 \\ 1 \times 1 + 1 \times -1 & 1 \times -2 + 1 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Use $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 0 \times 2 + -1 \times -1 \\ 1 \times 2 + 2 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Use $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 4 + 1 \times 1 \\ 1 \times 4 + 1 \times 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + -2 \times 0 & 1 \times 0 + -2 \times 1 \\ -1 \times 1 + 3 \times 0 & -1 \times 0 + 3 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{IB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 3 + 0 \times 1 & 1 \times 2 + 0 \times 1 \\ 0 \times 3 + 1 \times 1 & 0 \times 2 + 1 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 3 + -2 \times 1 & 1 \times 2 + -2 \times 1 \\ -1 \times 3 + 3 \times 1 & -1 \times 2 + 3 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 1 + 2 \times -1 & 3 \times -2 + 2 \times 3 \\ 1 \times 1 + 1 \times -1 & 1 \times -2 + 1 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + -2 \times -1 & 1 \times -2 + -2 \times 3 \\ -1 \times 1 + 3 \times -1 & -1 \times -2 + 3 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$$

$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 3 + 2 \times 1 & 3 \times 2 + 2 \times 1 \\ 1 \times 3 + 1 \times 1 & 1 \times 2 + 1 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Use $\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + -2 \times 0 & 1 \times -1 + -2 \times 1 \\ -1 \times 1 + 3 \times 0 & -1 \times -1 + 3 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

Use $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \times 2 + -8 \times 1 & 3 \times 1 + -8 \times 1 \\ -4 \times 2 + 11 \times 1 & -4 \times 1 + 11 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$$

2 a A product is defined only if the number of columns in the first matrix equals the number of rows of the second.

$\mathbf{A}$ has 2 columns and $\mathbf{Y}$ has 2 rows, so $\mathbf{AY}$ is defined.

$\mathbf{Y}$ has 1 column and $\mathbf{A}$ has 2 rows, so $\mathbf{YA}$ is not defined.

$\mathbf{X}$ has 1 column and $\mathbf{Y}$ has 2 rows, so $\mathbf{XY}$ is not defined.

$\mathbf{X}$ has 1 column and 2 rows, so $\mathbf{X}^2$ is not defined.

$\mathbf{C}$ has 2 columns and $\mathbf{I}$ has 2 rows, so $\mathbf{CI}$ is defined.

$\mathbf{X}$ has 1 column and $\mathbf{I}$ has 2 rows, so $\mathbf{XI}$ is not defined.

3 $$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \times 0 + 0 \times -3 & 2 \times 0 + 0 \times 2 \\ 0 \times 0 + 0 \times -3 & 0 \times 0 + 0 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4 $$\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

No, because $\mathbf{Q.2}$ part $\mathbf{b}$ shows that $\mathbf{AB}$ can equal $\mathbf{O}$, and $\mathbf{A} \neq \mathbf{O}, \mathbf{B} \neq \mathbf{O}$.

5 One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

6 $$\mathbf{LX} = [2 \ -1] \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$
$$= [2 \times 2 + -1 \times -3] = [7]$$
$$\mathbf{XL} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} [2 \ -1]$$
$$= \begin{bmatrix} 2 \times 2 & 2 \times -1 \\ -3 \times 2 & -3 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

7 A product is defined only if the number of columns in the first matrix equals the number of rows of the second. This can only happen if $m = n$, in which case both products will be defined.

8 $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a \times d + b \times -c & a \times -b + b \times a \\ c \times d + d \times -c & c \times -b + d \times a \end{bmatrix}$$
$$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the equations to be equal, all corresponding entries must be equal, therefore $ad - bc = 1$.

When written in reverse order, we get

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d\times a + -b\times c & d\times b + -b\times d \\ -c\times a + a\times c & -c\times b + a\times d \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad- \quad bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since $ad - bc = 1$.

9 We can use any values of a, b, c and d as long as $ad - bc = 1$.

For example, $a = 5, d = 2, b = 3, c = 3$ satisfy $ad - bc = 1$ and give

$$\mathbf{AB} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Other values could be chosen.

10 One possible answer.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1+0 & 2+1 \\ 4+2 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$$

$$\mathbf{B}+\mathbf{C} = \begin{bmatrix} 0+-1 & 1+2 \\ 2+-2 & 3+1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times -1 + 2\times 0 & 1\times 3 + 2\times 4 \\ 4\times -1 + 3\times 0 & 4\times 3 + 3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 0 + 2\times 2 & 1\times 1 + 2\times 3 \\ 4\times 0 + 3\times 2 & 4\times 1 + 3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix}$$

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times -1 + 2\times -2 & 1\times 2 + 2\times 1 \\ 4\times -1 + 3\times -2 & 4\times 2 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$\mathbf{AB}+\mathbf{AC} = \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4+-5 & 7+4 \\ 6+-10 & 13+11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$$

$$(\mathbf{B}+\mathbf{C})\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times 1 + 3\times 4 & -1\times 2 + 3\times 3 \\ 0\times 1 + 4\times 4 & 0\times 2 + 4\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

11 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

12a

$$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \times 1 + 12 \times 2 \\ 2.50 \times 1 + 3.00 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$$

$1 \times 5\,\mathbf{min}$ plus $2 \times 12\,\mathbf{min}$ means 29 min for one milkshake and two banana splits.

The total cost is $8.50.

b

$$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 \times 1 + 12 \times 2 & 5 \times 2 + 12 \times 1 & 5 \times 0 + 12 \times 1 \\ 2.5 \times 1 + 3 \times 2 & 2.5 \times 2 + 3 \times 1 & 2.5 \times 0 + 3 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$$

The matrix shows that John spent $29\,\mathbf{min}$ and $\$8.50$, one friend spent $22\,\mathbf{min}$ and $\$8.00$ (2 milkshakes and 1 banana split) while the other friend spent $12\,\mathbf{min}$ and $\$3.00$ (no milkshakes and 1 banana split).

13 $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$, $\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

14 $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$